NOTE ON MULTIPLY PERFECT NUMBERS

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The object of this note is to prove the following propostion :

There exist no multiply perfect numbers of multiplicity* 3 *of the form* $m = p_1^{a_1} p_2^{a_2} p_3^{2a_3}$ *where* p_1, p_2, p_3 *are distinct primes and* $p_1 < p_2 < p_3$.

In the Annals of Mathematics, ser. 2, vol. 2, p. 172 (July 1901), Dr. Jacob Westlund shows that $p_1 = 2$ and $p_2 = 3$. Hence we have (cf. Westlund, *l. c.*), supposing that there are such multiply perfect numbers,

$$(1) \qquad 3 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{3^{a_2+1} - 1}{3^{a_2} \cdot 2} \cdot \frac{p_3^{2a_3+1} - 1}{p_3^{2a_3}(p_3 - 1)},$$

from which we get

$$(2) \qquad p_3^{2a_3} + p_3^{2a_3-1} + \cdots + p_3 + 1 = \frac{p_3^{2a_3} \cdot 2^{a_1+1} \cdot 3^{a_2+1}}{(2^{a_1+1} - 1)(3^{a_2+1} - 1)}.$$

The first member is not divisible by p_3; hence,

$$(3) \qquad p_3^{2a_3} + p_3^{2a_3-1} + \cdots + p_3 + 1 = 2^\lambda 3^\mu,$$

where λ and μ are to be determined and belong to the series 0, 1, 2, 3 $\cdots$.

Since p_3 is a prime greater than 3 it is of one of the forms $6n \pm 1$.

Case I. Let $p_3 = 6n - 1$. Substituting in equation (3), we have

$$(4) \qquad (6n - 1)^{2a_3} + (6n - 1)^{2a_3-1} + \cdots + 1 = 2^\lambda 3^\mu.$$

If each term of the first member is expanded, the sum of the resulting terms not containing $6n$ as a factor is 1. Hence, this first number is divisible by neither 2 nor 3. This case therefore cannot yield multiply perfect numbers.

* This term was introduced by D. N. Lehmer in 1901; see Annals of Mathematics, ser. 2, vol. 2, p. 103: "a multiply perfect number is one which is an exact divisor of the sum of all the divisors, the quotient being the multiplicity."

Case II. Let $p_3 = 6n + 1$. Substituting in equation (3) after summing its left member, we have

$$\frac{(6n+1)^{2a_3+1}-1}{6n} = 2^\lambda 3^\mu, \tag{5}$$

whence

$$(6n)^{2a_3} + (2a_3+1)(6n)^{2a_3-1} + \cdots + a_3(2a_3+1)(6n) + (2a_3+1) = 2^\lambda 3^\mu. \tag{6}$$

Let $2a_3 + 1 = 3^\nu \cdot t$, where t contains neither 2 nor 3 as a factor. Each term but the last of the first member of (6) is divisible by $2 \cdot 3^{\nu-1}$. Hence, dividing through by 3^ν, equation (6) may be written (q being an integer quotient),

$$6 \cdot q + t = 2^\lambda 3^{\mu-\nu}. \tag{8}$$

The first member of (8) is greater than 1 and is divisible by neither 2 nor 3. Hence, equation (8) being impossible, case II yields no multiply perfect numbers.

Hence the proposition.